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Inequalities for some classical spin vector models

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Abstract. Inequalities are derived for a class of lattice systems including classical anisotropic $X-Y$ and Heisenberg ferromagnets. Inequalities relating different models are also established and we point out their connections with the existence problem of phase transitions.

1. Introduction

In the last few years correlation inequalities have been studied in several interesting directions in statistical mechanics and in constructive quantum field theory. We shall show in this paper how it is possible to combine the inequalities of Ginibre (1970) with those of Fortuin *et al* (1971) (see also Holley 1974, Preston 1974) in order to obtain new inequalities. We shall consider explicit models of ferromagnetism. An abstract version of our results appeared in Kunz *et al* (1975).

The models which we shall consider belong to the class of the classical spin vector models. They are defined on the lattice $\mathbb{Z}^d$, $d = 1, 2, \ldots$ as follows: to each $r \in \mathbb{Z}^d$ we associate a vector $S_r = (S_r^1, \ldots, S_r^D)$ of $\mathbb{R}^D$ with unit length. We call it a D-dimensional (classical) spin; it is parametrized by the points of the unit sphere in $\mathbb{R}^D$ and we choose for its distribution the normalized uniform measure on this sphere, written ds. The spins have ferromagnetic interactions. A typical Hamiltonian for the system defined on a finite subset Λ of $\mathbb{Z}^d$ is

$$-H = \sum_{\substack{i \neq j \\ i,j \in \Lambda}} J_{ij}^1 \Big(S_i^1 S_j^1 + \gamma \sum_{\alpha=2}^{D} S_i^\alpha S_j^\alpha\Big) + \sum_{i \in \Lambda} h_i S_i^1$$

with

$$J_{ij}^1 = J^1(|i-j|) \geqslant 0, \qquad |\gamma| \leqslant 1.$$

Here $|i-j|$ is the Euclidean distance between i and j. The constant γ describes some anisotropy in the interaction between two spins and h_i is an external inhomogeneous magnetic field. For $D = 1, 2$ or 3 we have respectively the Ising models, $X-Y$ models or Heisenberg models. We can apply our inequalities to obtain simply a number of rigorous results about phase transitions.

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(i) A simple proof of the absence of spontaneous magnetization for $D>2$ whenever this is true for $D=2$ (see Mermin 1967, Vuillermot and Romerio 1975).

(ii) A proof of the existence of a spontaneous magnetization for the model with $D=2$ and $d\geq 2$ if $|\gamma|<1$ at low enough temperatures (Kunz 1974, Malishev 1975).

(iii) A monotonicity property of the critical temperatures as a function of D

$$T_c(1)\geq T_c(2)\geq T_c(D), \qquad D\geq 3.$$

Here T_c is the temperature below which the spontaneous magnetization is non-zero. These inequalities were already empirically obtained from an analysis of series expansion (Stanley 1974).

2. General results

In this section we reveal our method from a general point of view and we shall consider particular examples in the subsequent sections. We begin with a short review of the known results about inequalities to be used extensively later. Our main result is formulated in proposition 4.

All models which we consider are defined on a finite subset Λ with N elements (see § 1). We do not write this explicitly. The Hamiltonian of a model for D-dimensional spins is

$$-H_D(\boldsymbol{J},\boldsymbol{h})=\sum_{i\neq j}\sum_{\alpha=1}^{D} J_{ij}^{\alpha}S_i^{\alpha}S_j^{\alpha}+\sum_{i}\sum_{\alpha=1}^{D} h_i^{\alpha}S_i^{\alpha}. \tag{2.1}$$

The interpretation of the parameters J_{ij}^{α} and h_i^{α} is as in § 1. We write $\boldsymbol{J}$ for $\{J^{\alpha}\}$, J^{α} for $\{J_{ij}^{\alpha}\}$, $J^{\alpha}\geq 0$ for $J_{ij}^{\alpha}\geq 0$ for all i, j and so on. Expectation values are computed with the usual Gibbs distribution defined from (2.1) and written $\langle\cdots\rangle_D(\boldsymbol{J},\boldsymbol{h})$ or $\langle\cdots\rangle_D$. Later we shall need the following results for $D=1$ and $D=2$ (Ginibre 1970).

(i) $D=1$. The spin S takes only two values ± 1 and its distribution is $\mathrm{d}s=\frac{1}{2}(\delta(S+1)+\delta(S-1))$. Let F_1 be the set of functions which can be expanded in powers of the functions $S_A=\Pi_{i\in A}S_i$, $A\subset\Lambda$, with positive coefficients only. For example if $J\geq 0$ and $h\geq 0$, $\exp(-H_1(J,h))\in F_1$.

Proposition 1. If $-H_1(J,h)\in F_1$ and $f, g\in F_1$, then

$$\langle f\rangle_1(J,h)\geq 0$$

$$\langle fg\rangle_1(J,h)\geq\langle f\rangle_1(J,h)\langle g\rangle_1(J,h).$$

Remarks. These inequalities were first proved by Griffiths (1967). The first inequality says, for example, that the magnetization is positive in a ferromagnetic Ising model defined by (2.1) with $J\geq 0$ and $h\geq 0$. The second inequality says that this magnetization increases, if we increase the ferromagnetic couplings $J\geq 0$ or the magnetic field h.

(ii) $D=2$. The spin S is described by $S=(\cos\theta, \sin\theta)$, $0\leq\theta\leq 2\pi$, and $\mathrm{d}s=(1/2\pi)\,\mathrm{d}\theta$. Let $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_N)$, $\theta_i\in[0,2\pi]$, $\boldsymbol{m}=(m_1,\ldots,m_N)$, $m_i\in\mathbb{Z}$ and $\boldsymbol{m}\cdot\boldsymbol{\theta}=\Sigma_{i=1}^{N} m_i\theta_i$. Let F_2 be the set of the functions which can be expanded in powers of $\cos\boldsymbol{m}\cdot\boldsymbol{\theta}$ with positive coefficients only. Examples of functions of F_2 are $\cos(\theta_i\pm\theta_j)$

($m_k = 0$, $k \neq i, j$, $m_i = 1$ and $m_j = \pm 1$) and $\exp(-H_2(\boldsymbol{J}, \boldsymbol{h}))$ if $J^1 \geqslant |J^2|$, $h^1 \geqslant 0$, $h^2 = 0$ ($H_2(\boldsymbol{J}, \boldsymbol{h})$ has an expansion in terms of $\cos(\theta_i \pm \theta_j)$ with positive coefficients).

Proposition 2. If $-H_2(\boldsymbol{J}, \boldsymbol{h}) \in F_2$ and $f, g \in F_2$ then

$$\langle f \rangle_2(\boldsymbol{J}, \boldsymbol{h}) \geqslant 0$$

$$\langle fg \rangle_2(\boldsymbol{J}, \boldsymbol{h}) \geqslant \langle f \rangle_2(\boldsymbol{J}, \boldsymbol{h}) \langle g \rangle_2(\boldsymbol{J}, \boldsymbol{h}).$$

The physical meaning of these inequalities is the same as for $D = 1$.

The starting-point of our method is to express a D-dimensional spin $S = (S^1, \ldots, S^D)$ by a family of two spins U and V of dimension D' and D'' with $D' + D'' = D$ and to use the results known for $D = 1$ and 2. This restricts the application of our method to $D \leqslant 4$, so long as we do not have a proposition like proposition 1 or 2 for $D > 2$. For the same reason in the case $D = 4$, we have no choice for the decomposition and we must take $D' = D'' = 2$. The D-dimensional spin S is also expressed as $S = (U', V')$ with U' and V' vectors of length $\cos^2\theta$ and $\sin^2\theta$ respectively and in $\mathbb{R}^{D'}$ and $\mathbb{R}^{D''}$ respectively. We rewrite this as follows

$$S = (\cos\theta U, \sin\theta V), \qquad 0 \leqslant \theta \leqslant \pi/2$$

with U and V unit vectors (=spins) in $\mathbb{R}^{D'}$ and $\mathbb{R}^{D''}$ respectively. We obtain for the spin distribution $\mathrm{d}s$ of one spin the factorization

$$\mathrm{d}s = \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\mu(\theta)$$

and for the Hamiltonian

$$H_D(\boldsymbol{J}, \boldsymbol{h}) = H_{D'}(\boldsymbol{J}', \boldsymbol{h}') + H_{D''}(\boldsymbol{J}'', \boldsymbol{h}'')$$

with

$$J'^{\alpha}_{ij} = J^{\alpha}_{ij} \cos\theta_i \cos\theta_j, \qquad h'^{\alpha}_i = h^{\alpha}_i \cos\theta_i, \qquad 1 \leqslant \alpha \leqslant D'$$

$$J''^{\alpha}_{ij} = J^{\alpha}_{ij} \sin\theta_i \sin\theta_j, \qquad h''^{\alpha}_i = h^{\alpha}_i \sin\theta_i, \qquad D'+1 \leqslant \alpha \leqslant D.$$

Example for $D = 2$. Here $D' = D'' = 1$ and we get

$$S = (\cos\theta U, \sin\theta V) \qquad 0 \leqslant \theta \leqslant \pi/2$$

$$\mathrm{d}s = \mathrm{d}u \, \mathrm{d}v \, (2/\pi) \, \mathrm{d}\theta.$$

The measure $\mathrm{d}u$ (or $\mathrm{d}v$) is the distribution of a spin of dimension one and thus $\mathrm{d}u = \frac{1}{2}(\delta(U+1) + \delta(U-1))$.

At this point we note that $\cos\theta$ and $\sin\theta$ are positive monotone functions on $[0, \pi/2]$. In order to elaborate this fact, we introduce a partial ordering on I which is the product of N copies of $[0, \pi/2]$: $\boldsymbol{\psi} \leqslant \boldsymbol{\Phi}$ if and only if $\psi_i \leqslant \Phi_i$, $\hat{\imath} = 1, \ldots, N$. It is easy to verify that I becomes a distributive lattice. In particular each pair of elements $\boldsymbol{\psi}$ and $\boldsymbol{\Phi}$ has a least upper bound $(\boldsymbol{\psi} \vee \boldsymbol{\Phi})_i = \max(\psi_i, \phi_i)$ and a greatest lower bound $(\boldsymbol{\psi} \wedge \boldsymbol{\Phi})_i = \min(\Psi_i, \phi_i)$. We say that a real function f in I is increasing if and only if $\boldsymbol{\psi} \leqslant \boldsymbol{\Phi} \Rightarrow f(\boldsymbol{\psi}) \leqslant f(\boldsymbol{\phi})$. We shall later have measures $\mathrm{d}\nu(\boldsymbol{\theta})$ of the following type:

$$\mathrm{d}\nu(\boldsymbol{\theta}) = p(\boldsymbol{\theta}) \prod_{i=1}^{N} \mathrm{d}\mu(\theta_i)$$

with $p(\boldsymbol{\theta}) \geqslant 0$ and satisfying

$$p(\boldsymbol{\psi} \vee \boldsymbol{\Phi}) p(\boldsymbol{\psi} \wedge \boldsymbol{\Phi}) \geqslant p(\boldsymbol{\psi}) p(\boldsymbol{\Phi}), \qquad \forall \boldsymbol{\Phi}, \boldsymbol{\psi} \in \dot{I}. \tag{2.2}$$

Such measures create the nice property (Preston 1974).

Proposition 3. Let $d\nu$ be a normalized measure on I defined as above. Then if f and g are both decreasing (increasing) on I

$$\int d\nu\, fg \geq \int d\nu\, f \int d\nu\, g.$$

If f is increasing and g decreasing, then the inequality is reversed.
Remarks. This proposition was first proved by Fortuin *et al* (1971) in the case of a discrete measure. In the case of an Ising model defined by (2.1) the Gibbs distribution has property (2.2) if $J \geq 0$.

We now are in a position to formulate and prove our main result.

Proposition 4. Let $D = 2, 3$ or 4. Let us suppose that $H_D(\boldsymbol{J}, \boldsymbol{h})$ can be decomposed as above with $-H_{D'}(\boldsymbol{J}', \boldsymbol{h}') \in F_{D'}$ and $-H_{D''}(\boldsymbol{J}'', \boldsymbol{h}'') \in F_{D''}$, $D', D'' = 1$ or 2. Let $f_i \in F_{D'}$, $i = 1, 2$, and $g \in F_{D''}$. Then

$$\langle h f_1 g \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \geq 0 \tag{2.3}$$

if h is a positive function on I,

$$\langle h_1 f_1 h_2 f_2 \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \geq \langle h_1 f_1 \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \langle h_2 f_2 \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \tag{2.4}$$

if h_i is positive monotone decreasing on I, $i = 1, 2$.

$$\langle h_1 f_1 k g \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \leq \langle h_1 f_1 \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \langle k g \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \tag{2.5}$$

if h_1 as above and k positive monotone increasing on I,

$$\langle h f_1 \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \leq \langle f_1 \rangle_{D'}(\boldsymbol{J}^*, \boldsymbol{h}^*) \langle h \rangle_D(\boldsymbol{J}, \boldsymbol{h}) \tag{2.6}$$

if h is positive on I and $J_{ij}^{*\alpha} = J_{ij}^{\alpha}$, $h_i^{*\alpha} = h_i^{\alpha}$, $1 \leq \alpha \leq D'$.

Proof. Let us begin with the proof of (2.3). We write $\langle f_1 g h \rangle_D$ using the decomposition of the Hamiltonian in two Hamiltonians for spins of lower dimensions. We get

$$\langle f_1 g h \rangle_D = \frac{\int \prod_{i=1}^N d\mu(\theta_i) Z_{D'}(\boldsymbol{\theta}) Z_{D''}(\boldsymbol{\theta}) \langle f_1 \rangle_{D'}(\boldsymbol{\theta}) \langle g \rangle_{D''}(\boldsymbol{\theta}) h(\boldsymbol{\theta})}{\int \prod_{i=1}^N d\mu(\theta_i) Z_{D'}(\boldsymbol{\theta}) Z_{D''}(\boldsymbol{\theta})}$$

which is straightforward and where $Z_{D'}$ is the partition function of the model with Hamiltonian $H_{D'}(\boldsymbol{J}', \boldsymbol{h}')$ and so on. We know that $\langle f_1 \rangle_{D'}(\boldsymbol{\theta})$ and $\langle g \rangle_{D''}(\boldsymbol{\theta})$ are positive functions by propositions 1 and 2 because $-H_{D'}$ and f_1 belong to $F_{D'}$ and $-H_{D''}$ and g belong to $F_{D''}$. This implies that $\langle f_1 g h \rangle_D \geq 0$. If we choose $g \equiv 1$, and apply the second inequality of proposition 1, which expresses a monotonicity property of $\langle h f_1 \rangle_{D'}$ as explained before, then we get

$$0 \leq \langle f_1 \rangle_{D'}(\boldsymbol{J}', \boldsymbol{h}')(\boldsymbol{\theta}) \leq \max_{\boldsymbol{\theta}} \langle f_1 \rangle_{D'}(\boldsymbol{J}', \boldsymbol{h}')(\boldsymbol{\theta}) = \langle f_1 \rangle_{D'}(\boldsymbol{J}^*, \boldsymbol{h}^*).$$

Now $\langle f_1 \rangle_{D'}(\boldsymbol{J}^*, \boldsymbol{h}^*)$ is independent of $\boldsymbol{\theta}$ and we have just proved (2.6).

The proofs of (2.4) and (2.5) are similar and we prove only (2.5). We begin as above and we write

$$\langle h_1 f_1 k g \rangle_D = \frac{\int \prod_{i=1}^N d\mu(\theta_i) Z_{D'}(\boldsymbol{\theta}) Z_{D''}(\boldsymbol{\theta}) h_1(\boldsymbol{\theta}) \langle f_1 \rangle_{D'}(\boldsymbol{\theta}) k(\boldsymbol{\theta}) \langle g \rangle_{D''}(\boldsymbol{\theta})}{\int \prod_{i=1}^N d\mu(\theta_i) Z_{D'}(\boldsymbol{\theta}) Z_{D''}(\boldsymbol{\theta})}.$$

Again applying propositions 1 or 2 we know that $f_1(\boldsymbol{\theta})$ is a positive decreasing function of $\boldsymbol{\theta}$, because $\cos\theta$ and consequently the coupling constants of the Hamiltonian $-H_{D'}$

are positive decreasing functions of $\boldsymbol{\theta}$. In the same manner $g(\boldsymbol{\theta})$ is a positive increasing function. On the other hand we can write

$$\langle h_1 f_1 k g\rangle_D = \int \mathrm{d}\nu\, h_1 f_1 k g$$

with

$$p(\boldsymbol{\theta}) = \frac{Z_{D'}(\boldsymbol{\theta}) Z_{D''}(\boldsymbol{\theta})}{\int \Pi_{i=1}^{N} \mathrm{d}\mu(\theta_i) Z_{D'}(\boldsymbol{\theta}) Z_{D''}(\boldsymbol{\theta})}.$$

Let us suppose for an instant that $p(\boldsymbol{\theta})$ satisfies the property (2.2) which is required for the application of proposition 3. Then we obtain the desired result by applying this proposition to the functions $h_1 f_1$ and kg. It remains thus to check the property (2.2) for the function $p(\boldsymbol{\theta})$. Clearly we can drop the normalization factor and we just have to prove that

$$Z_{D'}(\boldsymbol{\psi} \vee \boldsymbol{\Phi}) Z_{D'}(\boldsymbol{\psi} \wedge \boldsymbol{\Phi}) \geqslant Z_{D'}(\boldsymbol{\psi}) Z_{D'}(\boldsymbol{\Phi}) \qquad \text{with } D' = 1 \text{ or } 2. \tag{2.7}$$

Let us take $D' = 1$. A similar proof holds for $D' = 2$ or $D'' = 1, 2$. Consequently we drop the index 1 from now on. First we prove a property about positive monotone functions which implies, together with proposition 1, the desired property for $p(\boldsymbol{\theta})$.

Lemma. Let q_i be a positive monotone decreasing function $[0, \pi/2]$, $i = 1, \ldots, N$. Then for all $\boldsymbol{\psi}$ and $\boldsymbol{\Phi} \in I$ and $B \subset \Lambda$

$$f_B(\boldsymbol{\Phi} \vee \boldsymbol{\psi}) + f_B(\boldsymbol{\Phi} \wedge \boldsymbol{\psi}) \geqslant f_B(\boldsymbol{\Phi}) + f_B(\boldsymbol{\psi})$$

with $f_B = \Pi_{i \in B} f_i$. The lemma is also true, if the q_i are positive monotone decreasing on $[0, \pi/2]$.

Proof. Let us put $f_B(\boldsymbol{\Phi} \wedge \boldsymbol{\psi}) = a$, $f(\boldsymbol{\Phi} \vee \boldsymbol{\psi}) = b$, $f(\boldsymbol{\Phi}) = c$ and $f(\boldsymbol{\psi}) = d$. By hypothesis a, b, c and d are non-negative numbers with $a \geqslant c$, $a \geqslant d$ and $ab = cd$. These four numbers satisfy $a + b \geqslant c + d$ which proves the lemma. (See lemma 2 in Preston 1974).

We now prove (2.7) which is equivalent to

$$\frac{Z(\boldsymbol{\Phi} \wedge \boldsymbol{\psi})}{Z(\boldsymbol{\Phi})} \Big/ \frac{Z(\boldsymbol{\psi})}{Z(\boldsymbol{\Phi} \vee \boldsymbol{\psi})} \geqslant 1.$$

The left-hand side of this last expression is

$$\frac{\langle \exp[-\beta H(J'(\boldsymbol{\Phi} \wedge \boldsymbol{\psi}) - J'(\boldsymbol{\Phi}), h'(\boldsymbol{\Phi} \wedge \boldsymbol{\psi}) - h'(\boldsymbol{\phi}))]\rangle (J'(\boldsymbol{\Phi}), h'(\boldsymbol{\Phi}))}{\langle \exp[-\beta H(J'(\boldsymbol{\psi}) - J'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}), h'(\boldsymbol{\psi}) - h'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}))]\rangle (J'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}), h'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}))}$$

β being the inverse temperature. Using the lemma we get

$$J'(\boldsymbol{\Phi} \wedge \boldsymbol{\psi}) - J'(\boldsymbol{\Phi}) \geqslant J'(\boldsymbol{\psi}) - J'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}) \geqslant 0 \tag{2.8}$$

and a similar expression for $h'(\boldsymbol{\Phi})$. This implies that $\exp[\ldots]$ belongs to F_1 and we obtain the following lower bound

$$\langle \exp[-\beta H(J'(\boldsymbol{\Phi} \wedge \boldsymbol{\psi}) - J'(\boldsymbol{\Phi}), h'(\boldsymbol{\Phi} \wedge \boldsymbol{\psi}) - h'(\boldsymbol{\phi}))]\rangle (J'(\boldsymbol{\Phi}), h'(\boldsymbol{\Phi})) \geqslant$$
$$\langle \exp[-\beta H(J'(\boldsymbol{\psi}) - J'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}), h'(\boldsymbol{\psi}) - h'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}))]\rangle (J'(\boldsymbol{\Phi}), h'(\boldsymbol{\Phi})) \geqslant$$
$$\langle \exp[-\beta H(J'(\boldsymbol{\psi}) - J'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}), h'(\boldsymbol{\psi}) - h'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}))]\rangle (J'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}), h'(\boldsymbol{\Phi} \vee \boldsymbol{\psi}))$$

by applying first (2.8) and then proposition 1. This lower bound implies (2.7) and thus (2.5) is proved.

Remarks. In the proof of proposition 4 we used only propositions 1 to 3. We can consider more general Hamiltonians than (2.1). In particular we can introduce three-body interactions and so on. The only property which is required is that $-H_{D'} \in F_{D'}$ and $-H_{D''} \in F_{D''}$. We may also take other single-spin distributions than ds for the D-dimensional spin S. The only property which we need is that the distribution ds satisfies $ds = du\, dv\, d\mu(\theta)$ with some probability measure $d\mu(\theta)$, du and dv being defined as above.

3. Examples

In this section we consider examples where we can apply our method. The different results which we obtain are simply a translation of proposition 4 in particular situations. Therefore we do not give further details. At the end of the section we derive one more result valid for all D which implies the announced results (i) and (iii) of § 1. We begin with $D = 2$.

3.1. $D = 2$

In this case $H_D(\boldsymbol{J}, \boldsymbol{h})$ becomes

$$-H_2(J^1, J^2, h^1, h^2) = \sum_{i \neq j} (J^1_{ij} \cos\theta_i \cos\theta_j + J^2_{ij} \sin\theta_i \sin\theta_j) + \sum_i (h^1_i \cos\theta_i + h^2_i \sin\theta_i). \tag{3.1}$$

The decomposition of a spin $S = (S^1, S^2) = (\cos\theta U, \sin\theta V)$ has been treated as an example in the last section. In this way for $H_{D'}(J'^1, h'^1)$ and $H_{D''}(J''^2, h''^2)$ we get

$$-H_{D'=1}(J'^1, h'^1) = \sum_{i \neq j} J^1_{ij} \cos\theta_i \cos\theta_j U_i U_j + \sum_i h^1_i \cos\theta_i U_i$$

and

$$-H_{D''=1}(J''^2, h''^2) = \sum_{i \neq j} J^2_{ij} \sin\theta_i \sin\theta_j V_i V_j + \sum_i h^2_i \sin\theta_i V_i$$

where $\theta_i \in [0, \pi/4]$ and U_i and V_i are now one-dimensional spins. Applying proposition 4 we obtain proposition 5.

Proposition 5. Let us consider a model defined by (3.1) with $J^1 \geq 0$, $J^2 \geq 0$, $h^1 \geq 0$ and $h^2 \geq 0$. Then we have

$$\langle S^i_A S^i_B \rangle \geq \langle S^i_A \rangle \langle S^i_B \rangle, \qquad i = 1, 2$$

$$\langle S^1_A S^2_B \rangle \leq \langle S^1_A \rangle \langle S^2_B \rangle$$

$$\langle S^1_A S^2_B \rangle \geq 0.$$

$$\langle S^1_A \rangle_2 (J^1, J^2, h^1, h^2) \leq \langle U_A \rangle_1 (J^1, h^1) \equiv \langle S^1_A \rangle_1 (J^1, h^1)$$

$$\langle S_i \cdot S_j \rangle \geq \langle S_i \rangle \cdot \langle S_j \rangle$$

where $S^1_A = \Pi_{i \in A} S^1_i$, $A \subset \Lambda$, and the dot $\cdot$ means the scalar product.

Remarks. (i) The first four inequalities follow respectively from (2.4), (2.5), (2.3) and (2.6) of proposition 4. The last inequality is an immediate consequence of the first one.

(ii) These inequalities have a simple physical meaning. Let us consider a model defined by $-H_2(J^1, J^2, h, 0)$, $J^1 \geqslant 0$, $J^2 \geqslant 0$, $h \geqslant 0$ and for example the magnetization

$$m_\Lambda(J^1, J^2, h) = \frac{1}{|\Lambda|} \sum_{i\in\Lambda} \langle S_i^1 \rangle_2 (J^1, J^2, h) \tag{3.2}$$

where $|\Lambda|$ is the number of elements of the set Λ on which the model is defined. It is easy to see that the inequalities of proposition 5 imply that the magnetization is a positive function which increases if h or J^1 increase but which decreases if J^2 increases. These properties are of course also valid whenever the thermodynamic limit of (3.2) exists. (iii) We have noted already that we can take more general interactions. For example

$$-H_2 = \sum_{\phi \neq A \subset \Lambda} J_A^1 S_A^1 + J_A^2 S_A^2 + \sum_{i\in\Lambda} \lambda_i (S_i^1)^2$$

with $J_A^i \geqslant 0$, $i = 1, 2$ and any λ_i. The λ_i describe an inhomogeneous crystal field. (iv) Recently Monroe (1975) proved part of proposition 5 for two-body interactions using a different technique. Bricmont (1975) also proved proposition 5 by yet another method. (v) We can derive other inequalities by using the rotational symmetry of the interactions if $J^1 = J^2$. In this case we can write (3.1) as follows

$$-H_2(\boldsymbol{J}, \boldsymbol{h}) = -H_2(J, J, h^1, h^2) = \sum_{i\neq j} J_{ij} S_i \cdot S_j + \sum_i h_i \cdot S_i. \tag{3.3}$$

If we perform a rotation R about $\pi/4$, a spin $S = (S^1, S^2)$ becomes $T = RS = (T^1, T^2) = \frac{1}{2}\sqrt{2}((S^1 - S^2), (S^1 + S^2))$. Using $RS_i \,.\, RS_j = S_i \,.\, S_j$ and the invariance of ds under rotation, we obtain for example the relation

$$\langle (S^1 + S^2)_A \rangle_2 (J, J, h^1, h^2) = (\sqrt{2})^{|A|} \langle T_A^2 \rangle_2 (J, J, \tfrac{1}{2}\sqrt{2}(h^1 - h^2), \tfrac{1}{2}\sqrt{2}(h^1 + h^2)).$$

By this method we immediately get proposition 6 from proposition 5.

Proposition 6. Let us consider a model defined by (3.3) with $J_{ij} = J_{ij}^1 = J_{ij}^2 \geqslant 0$, $h^1 \geqslant |h^2|$. Then we have

$$\langle (S^1 + S^2)_A (S^1 + S^2)_B \rangle_2 \geqslant \langle (S^1 + S^2)_A \rangle_2 \langle (S^1 + S^2)_B \rangle_2$$
$$\langle (S^1 - S^2)_A (S^1 - S^2)_B \rangle_2 \geqslant \langle (S^1 - S^2)_A \rangle_2 \langle (S^1 - S^2)_B \rangle_2$$
$$\langle (S^1 - S^2)_A (S^1 + S^2)_B \rangle_2 \leqslant \langle (S^1 - S^2)_A \rangle_2 \langle (S^1 + S^2)_B \rangle_2.$$

Corollary 7. Under the conditions of proposition 6 we have

$$\langle S_i^1 S_j^1 \rangle_2 - \langle S_i^1 \rangle_2 \langle S_j^1 \rangle_2 \leqslant \langle S_i^2 S_j^2 \rangle_2 - \langle S_i^2 \rangle_2 \langle S_j^2 \rangle_2$$
$$(\langle S_i^1 S_j^1 \rangle_2 - \langle S_i^1 \rangle_2 \langle S_j^1 \rangle_2) + (\langle S_i^2 S_j^2 \rangle_2 - \langle S_i^2 \rangle_2 \langle S_j^2 \rangle_2)$$
$$\geqslant (\langle S_i^1 S_j^2 \rangle_2 - \langle S_i^1 \rangle_2 \langle S_j^2 \rangle_2) + (\langle S_j^1 S_i^2 \rangle_2 - \langle S_j^1 \rangle_2 \langle S_i^2 \rangle_2).$$

Remarks. Let us consider the model defined by (3.3) with $J \geqslant 0$, $h_1 = h \geqslant 0$ and $h_2 = 0$. The parallel susceptibility $\chi^{\parallel}$ is the derivative of the magnetization with respect to the magnetic field h (see (3.2)), i.e.

$$\chi_\Lambda^{\parallel} = \frac{\partial m_\Lambda}{\partial h} = kT \frac{1}{|\Lambda|} \sum_{i,j\in\Lambda} (\langle S_i^1 S_j^1 \rangle - \langle S_i^1 \rangle \langle S_j^1 \rangle).$$

Here T stands for the temperature and k is the Boltzmann's constant. One also usually defines a transverse susceptibility $\chi^\perp$ by

$$\chi_\Lambda^\perp = kT\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}(\langle S_i^2 S_j^2\rangle - \langle S_i^2\rangle\langle S_j^2\rangle).$$

We thus obtain from the first inequality of the corollary that the parallel susceptibility is dominated by the transverse susceptibility. Both are positive by the first inequality of proposition 5. These properties are also true at the thermodynamical limit.

3.2. $D = 3$

In this case $D' = 1$ and $D'' = 2$, and

$$ds = du\, dv \sin\theta\, d\theta.$$

It is important to realize in this case that the decomposition of $S = (S^1, \ldots, S^3) = (\cos\theta U, \sin\theta V)$ is not the only possible one. We can, for example, collect the first and the third components of S in order to form another two-dimensional spin and so forth. There is also no unique choice of the conditions on $\boldsymbol{J}$ and $\boldsymbol{h}$. We consider only two cases. For example we have propositions 8 and 9.

Proposition 8. Let us consider a model defined by (2.1) with $D = 3$, $J_{ij}^1 \geq J_{ij}^2 \geq J_{ij}^3 \geq 0$, $h_i^1 \geq 0$ and $h_i^2 = h_i^3 = 0$. Then the following is true:

$$\langle S_A^k S_B^k\rangle_3 \geq \langle S_A^k\rangle_3\langle S_B^k\rangle_3 \qquad k = 1, 2, 3$$

$$\langle S_A^1 S_B^k\rangle_3 \leq \langle S_A^1\rangle_3\langle S_B^k\rangle_3 \qquad k = 2, 3$$

$$\langle S_A^1 S_B^2 S_C^3\rangle_3 \geq 0$$

$$\langle S_A^1\rangle_3(J^1, J^2, J^3, h^1) \leq \langle S_A^1\rangle_2(J^1, J^2, h^1)$$

$$\langle S_1 \cdot S_j\rangle_3 \geq \langle S_i\rangle_3 \cdot \langle S_j\rangle_3.$$

On the other hand, if we have isotropic two-body interactions we can also derive new inequalities by using the method of proving proposition 6.

Proposition 9. Let us consider a model defined by (2.1) with $D = 3$, $J_{ij}^1 = J_{ij}^2 = J_{ij}^3 \geq 0$, $h_i^1 \geq 0$ and $h_i^2 = h_i^3 = 0$. Then the following is true:

$$\langle (S^1 + S^k)_A (S^1 + S^k)_B\rangle_3 \geq \langle (S^1 + S^k)_A\rangle_3 \langle (S^1 + S^k)_B\rangle_3 \qquad k = 2, 3$$

$$\langle (S^1 - S^k)_A (S^1 - S^k)_B\rangle_3 \geq \langle (S^1 - S^k)_A\rangle_3 \langle (S^1 - S^k)_B\rangle_3 \qquad k = 2, 3$$

$$\langle (S^1 - S^k)_A (S^1 + S^k)_B\rangle_3 \leq \langle (S^1 - S^k)_A\rangle_3 \langle (S^1 + S^k)_B\rangle_3 \qquad k = 2, 3$$

and in particular

$$\langle S_i^1 S_j^1\rangle_3 - \langle S_i^1\rangle_3\langle S_j^1\rangle_3 \leq \langle S_i^k S_j^k\rangle_3 \qquad k = 2, 3$$

Remark. The interpretation of these two propositions is like that of propositions 5 and 6 and corollary 7.

3.3. $D = 4$

Here $D' = D'' = 2$ and

$$ds = du\, dv\, 2\sin\theta\cos\theta\, d\theta.$$

We can make the same remarks as above. We obtain for example the analogue of proposition 5 with $J^1 = J^2 \geqslant J^3 = J^4 \geqslant 0$ and $h^1 \geqslant 0$, $h^2 = h^3 = h^4 = 0$.

3.4.

If the dimension of the spin is bigger than 4, we cannot apply our method. However we can prove the following result for $D > 3$.

Proposition 10. Let us consider the model defined by (2.1) with $J^1 \geqslant J^2 \geqslant 0$, $h^1 \geqslant 0$, $h^2 = \ldots = h^D = 0$ for any $D \geqslant 3$. Then we have

$$\langle S_A^1\rangle_1(J^1, h^1) \geqslant \langle S_A^1\rangle_2(J^1, J^2, h^1) \geqslant \langle S_A^1\rangle_D(J^1, \ldots, J^D, h^1) \tag{3.4}$$

Proof. The first inequality is a result of proposition 5. Let $D \geqslant 3$. We decompose the spin S with $D' = 2$ and $D'' = D - 2$. The proof then reduces to that of (2.6) of proposition 4.

Remarks. (i) Let us consider isotropic ferromagnetic models characterized by $D, J > 0$ and $h \geqslant 0$. By convention we choose the magnetic field $(h, 0, \ldots, 0)$. We take the thermodynamical limit and denote the magnetization by $m(D, J, h)$. Using proposition 10 we get

$$m(1, J, h) \geqslant m(2, J, h) \geqslant m(D, J, h) \geqslant 0, \qquad D \geqslant 3.$$

The spontaneous magnetization is by definition

$$\lim_{h \to 0+} m(D, J, h) \equiv m^+(D, J).$$

Thus we have

$$m^+(1, J) \geqslant m^+(2, J) \geqslant m^+(D, J) \geqslant 0, \qquad D \geqslant 3. \tag{3.5}$$

The two results (i) and (iii) of § 1 then follow directly from (3.5).

(ii) The inequalities (3.4) are to be expected because the phase space of the spin S becomes larger with D. For isotropic systems we expect

$$\langle S_i \cdot S_j\rangle_D \geqslant \langle S_i \cdot S_j\rangle_{D+1}.$$

4. Lower bounds for models with $D = 2$

In this last section we compare the magnetizations of an anisotropic X–Y model and of an Ising model. The result gives a simple proof of the existence of a first-order phase transition of an anisotropic ferromagnetic X–Y model in two or more dimensions. The proof is based on our method of § 2 and a judicious choice of the reference frame inspired by our remarks of § 3 on the possibility of performing a rotation of the spins. We start with the model defined by (3.1) and we choose $J^1 \geqslant |J^2| \geqslant 0$, $h^1 \geqslant 0$, $h^2 = 0$. It is convenient to write $J^1 = J$, $J^2 = \gamma J$ and (3.1) as follows:

$$-H_2(J, \gamma, h) = \sum_{i \neq j} J_{ij}[\tfrac{1}{2}(1 + \gamma_{ij})](S_i^1 S_j^1 + S_i^2 S_j^2)$$
$$+ \sum_{i \neq j} J_{ij}[\tfrac{1}{2}(1 - \gamma_{ij})](S_i^1 S_j^1 - S_i^2 S_j^2) + \sum_i h_i S_i^1. \tag{4.1}$$

Proposition 11. Let us consider the model defined by (4.1) and $J \geqslant 0$, $h \geqslant 0$ and $|\gamma| < 1$. Then for all $A \subset \Lambda$

$$\langle S_A^1 \rangle_2(J, \gamma, h) \geqslant (\tfrac{1}{2})^{|A|} \langle S_A^1 \rangle_1[\tfrac{1}{4}J(1-|\gamma|), \tfrac{1}{2}h]$$

where $|A|$ is the number of elements of the set A.

Remarks. (i) From the proposition we obtain for the magnetization the relation

$$m(2, J, \gamma, h) \geqslant \tfrac{1}{2} m[1, \tfrac{1}{4}J(1-|\gamma|), \tfrac{1}{2}h].$$

Therefore the spontaneous magnetization satisfies

$$m^+(2, J, \gamma) \geqslant \tfrac{1}{2} m^+[1, \tfrac{1}{4}J, (1-|\gamma|)].$$

But the spontaneous magnetization of an Ising model is non-zero at low enough temperature if $d \geqslant 2$. We have thus proved result (ii) of the introduction.

(ii) Proposition 11 was first proved by Kunz (1974) using a different method in order to prove result (ii).

Proof. In this proposition we consider the projection of the spin along the direction given by the magnetic field. We do not apply our method directly. First we mix the two components of the spin by choosing a new reference frame in such a way that the magnetic field is now $(\tfrac{1}{2}\sqrt{2}h, \tfrac{1}{2}\sqrt{2}h)$; the new components of the spin are (T^1, T^2) and

$$-H_2(J, \gamma, h) = \sum_{i \neq j} J_{ij}[\tfrac{1}{2}(1+\gamma_{ij})](T_i^1 T_j^1 + T_i^2 T_j^2) + \sum_{i \neq j} J_{ij}[\tfrac{1}{2}(1-\gamma_{ij})](T_i^1 T_j^2 + T_i^2 T_j^1) + \sum_i \tfrac{1}{2}\sqrt{2} h_i (T_i^1 + T_i^2).$$

In the first step we used the fact that ds is invariant under rotation. In the next step we apply our method. But now the Hamiltonian $H_2(J, \gamma, h)$ does not split into two Hamiltonians of the Ising type which are coupled only by the variables θ as before. Owing to our choice of the reference frame we also get ferromagnetic couplings between the two Ising systems occurring in the decomposition or, in other words, only one Ising system with spins U_i and V_i and with Hamiltonian

$$\begin{aligned} &-\sum_{i \neq j} J_{ij}[\tfrac{1}{2}(1+\gamma_{ij})](\cos\theta_i \cos\theta_j U_i U_j + \sin\theta_i \sin\theta_j V_i V_j) \\ &\quad - \sum_{i \neq j} J_{ij}[\tfrac{1}{2}(1-\gamma_{ij})](\cos\theta_i \sin\theta_j U_i V_j + \sin\theta_i \cos\theta_j V_i U_j) \\ &\quad - \sum_i h_i \tfrac{1}{2}\sqrt{2}(\cos\theta_i U_i + \sin\theta_i V_i). \end{aligned} \tag{4.2}$$

We estimate $\langle (\tfrac{1}{2}\sqrt{2}(T^1+T^2))_A \rangle_2(J, \gamma, h)$. We have

$$\langle (T^1+T^2)_A \rangle_2 = \frac{\int \prod_{i=1}^N \mathrm{d}\mu(\theta_i) Z_1 \langle \prod_{i \in A} (\cos\theta_i U_i + \sin\theta_i V_i) \rangle_1}{\int \prod_{i=1}^N \mathrm{d}\mu(\theta_i) Z_1}$$

where $\mathrm{d}\mu(\theta_i) = (2/\pi)\,\mathrm{d}\theta_i$, $\langle \cdot \rangle_1$ is the average value computed with the Hamiltonian (4.2) and Z_1 is the corresponding partition function. We now note that $1 \pm \gamma_{ij} \geqslant 1 - |\gamma_{ij}|$ so that applying proposition 1 gives:

$$\left\langle \prod_{i \in A} (\cos\theta_i U_i + \sin\theta_i V_i) \right\rangle_1 \geqslant \left\langle \prod_{i \in A} (\cos\theta_i U_i + \sin\theta_i V_i) \right\rangle_1^+ \equiv C(\theta_i, \ldots, \theta_N)$$

where $\langle\cdot\rangle_1^+$ is the average value computed with the Hamiltonian given by (4.2) when we replace $1\pm\gamma_{ij}$ by $1-|\gamma_{ij}|$. Now we note that the function $C(\theta_1,\ldots,\theta_N)$ is invariant, if we change θ_i in $\pi/2-\theta_i$. This implies that

$$\min_{\substack{\theta_1,\ldots,\theta_N\\ 0\leqslant\theta_i\leqslant\pi/2}} C(\theta_1,\ldots,\theta_N)=\min_{\substack{\theta,\ldots,\theta_N\\ 0\leqslant\theta_i\leqslant\pi/4}} C(\theta_1,\ldots,\theta_N).$$

But on the interval $[0,\pi/4]$, $\cos\theta\geqslant\frac{1}{2}\sqrt{2}$ and $\sin\theta\geqslant 0$. By applying proposition 1 again, we get:

$$C(\theta_1,\ldots,\theta_N)\geqslant(\tfrac{1}{2}\sqrt{2})^{|A|}\langle S_A^1\rangle_1[\tfrac{1}{4}J(1-|\gamma|),\tfrac{1}{2}h]$$

where now the average value is computed with

$$-H_1[\tfrac{1}{4}J(1-|\gamma|),\tfrac{1}{2}h]=\sum_{i\neq j}J_{ij}[\tfrac{1}{4}(1-|\gamma_{ij}|)]S_i^1S_j^1+\sum_i\tfrac{1}{2}h_iS_i^1.$$

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